HARDY'S THEOREM FOR GABOR TRANSFORM

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Abstract. We establish analogues of Hardy's theorem for Gabor transform on locally compact abelian groups, Euclidean motion group and several general classes of nilpotent Lie groups which include Heisenberg groups, thread-like nilpotent Lie groups, 2-NPC nilpotent Lie groups and low-dimensional nilpotent Lie groups.

1. Introduction

The Hardy's uncertainty principle states that a non-zero integrable function f on $\mathbb{R}$ and its Fourier transform $\widehat{f}$ cannot both be compactly supported. For $f \in L^2(\mathbb{R})$, the Fourier transform $\widehat{f}$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\ e^{-2\pi i \xi x}\ dx.$$

The following theorem of Hardy (see [9] for proof) makes the above statement more precise:

Theorem 1.1. Let f be a measurable function on $\mathbb{R}$ such that

(i) $|f(x)| \leq Ce^{-a\pi x^2}$, for all $x \in \mathbb{R}$,
(ii) $|\widehat{f}(\xi)| \leq Ce^{-b\pi \xi^2}$, for all $\xi \in \mathbb{R}$,

where a, b and C are positive constants. If $ab > 1$, then $f = 0$ a.e.

The Hardy's theorem has been proved for Fourier transform in the setting of $\mathbb{R}^n$ and Heisenberg group $\mathbb{H}_n$ (see [17]), locally compact abelian groups and some classes of solvable Lie groups (see [1]), Euclidean motion group (see [16, 18]), nilpotent Lie groups (see [2, 10, 13, 14, 19]) and non-compact connected semisimple Lie groups with finite centre (see [15]). For detailed survey of uncertainty principles for Fourier transform, refer to [7].
Over the last many decades the Fourier transform was an indispensable tool in applied mathematics, especially in signal processing. It has also been recognised that global Fourier transform is of little practical value in analyzing the frequency spectrum of a long signal. So there is a necessity of the notion of frequency analysis that is local in time, in other words, a *joint time-frequency analysis*. In recent times, *Gabor transform* is one of the tools

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that has established itself in this direction. The approach used in this technique is cutting the signal into segments using a smooth window-function (usually square integrable function) and then computing the Fourier transform separately on each smaller segment. It results in a two-dimensional representation of the signal.

Let $\psi \in L^2(\mathbb{R})$ be a fixed function usually called a *window function*. The Gabor transform of a function $f \in L^2(\mathbb{R})$ with respect to the window function ψ is defined by

$$G_\psi f : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$$

such that

$$G_\psi f(t,\xi) = \int_{\mathbb{R}} f(x)\ \overline{\psi(x-t)}\ e^{-2\pi i \xi x}\ dx,$$

for all $(t,\xi) \in \mathbb{R} \times \widehat{\mathbb{R}}$.

The Hardy's theorem for Gabor transform on $\mathbb{R}^n$ has been established in [8]. The continuous Gabor transform for second countable, unimodular and type I group has been defined in [6], a brief description of which is given in section 2. In section 3, the Hardy's theorem for Gabor transform on a second countable, locally compact, abelian group has been established. An analogue of Hardy's theorem for Gabor transform on Euclidean motion group has been proved in the next section. In the last section, we shall prove Hardy's theorem for Gabor transform on several general classes of nilpotent Lie groups for which $\|\pi_\xi(f)\|_{\mathrm{HS}}$ has a particular form. These include Heisenberg groups, thread-like nilpotent Lie groups, 2-NPC nilpotent Lie groups and several low-dimensional nilpotent Lie groups.

2. Continuous Gabor Transform

Let G be a second countable, unimodular group of type I. Let dx denote the Haar measure on G and $d\pi$ the Plancherel measure on $\widehat{G}$. For each $(x,\pi) \in G \times \widehat{G}$, we define

$$\mathcal{H}_{(x,\pi)} = \pi(x)\mathrm{HS}(\mathcal{H}_\pi),$$

where $\pi(x)\mathrm{HS}(\mathcal{H}_\pi) = \{\pi(x)T : T \in \mathrm{HS}(\mathcal{H}_\pi)\}$. It can be easily seen that $\mathcal{H}_{(x,\pi)}$ forms a Hilbert space with the inner product given by

$$\langle \pi(x)T, \pi(x)S \rangle_{\mathcal{H}_{(x,\pi)}} = \mathrm{tr}\,(S^*T) = \langle T, S \rangle_{\mathrm{HS}(\mathcal{H}_\pi)}.$$

Also, $\mathcal{H}_{(x,\pi)} = \mathrm{HS}(\mathcal{H}_\pi)$ for all $(x,\pi) \in G \times \widehat{G}$. The family $\{\mathcal{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ of Hilbert spaces indexed by $G \times \widehat{G}$ is a field of Hilbert spaces over $G \times \widehat{G}$. Let $\mathcal{H}^2(G \times \widehat{G})$ denote the direct integral of $\{\mathcal{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ with respect to the product measure $dx\ d\pi$, i.e., the space of all measurable vector fields F on $G \times \widehat{G}$ such that

$$\|F\|^2_{\mathcal{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \|F(x,\pi)\|^2_{(x,\pi)}\ dx\ d\pi < \infty.$$

One can observe that $\mathcal{H}^2(G\times\widehat{G})$ forms a Hilbert space with the inner product given by

$$\langle F, K\rangle_{\mathcal{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \mathrm{tr}\,[F(x,\pi)K(x,\pi)^*]\; dx\; d\pi.$$

$C_c(G)$ will denote the set of all continuous complex-valued functions on G with compact supports. Let $f \in C_c(G)$ and ψ be a fixed function in $L^2(G)$. For $(x,\pi) \in G \times \widehat{G}$, the continuous *Gabor Transform* of f with respect to the window function ψ can be defined as a measurable field of operators on $G \times \widehat{G}$ by

$$G_\psi f(x,\pi) := \int_G f(y)\; \overline{\psi(x^{-1}y)}\; \pi(y)^*\; dy. \tag{2.1}$$

The operator-valued integral (2.1) is considered in the weak-sense, i.e., for each $(x,\pi) \in G \times \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$, we have

$$\langle G_\psi f(x,\pi)\xi, \eta\rangle = \int_G f(y)\; \overline{\psi(x^{-1}y)}\; \langle \pi(y)^*\xi, \eta\rangle\; dy.$$

One can verify that $G_\psi f(x,\pi)$ is a Hilbert-Schmidt operator for all $x \in G$ and for almost all $\pi \in \widehat{G}$. As in [6], for $f \in C_c(G)$ and a window function $\psi \in L^2(G)$, we have

$$\|G_\psi f\|_{\mathcal{H}^2(G\times\widehat{G})} = \|\psi\|_2\; \|f\|_2. \tag{2.2}$$

It means that the continuous Gabor transform $G_\psi : C_c(G) \to \mathcal{H}^2(G \times \widehat{G})$ defined by $f \mapsto G_\psi f$ is a multiple of an isometry. So, we can extend G_ψ uniquely to a bounded linear operator from $L^2(G)$ into a closed subspace H of $\mathcal{H}^2(G \times \widehat{G})$ which we still denote by G_ψ and this extension satisfies (2.2) for each $f \in L^2(G)$.

3. Locally Compact Abelian Groups

Throughout this section, we shall consider G to be a second countable, locally compact, abelian group and $\widehat{G}$ the dual group of G. For $z \in G$ and $\omega \in \widehat{G}$, we define the *translation operator* T_z on $L^2(G)$ as

$$(T_z f)(y) = f(z^{-1}y)$$

and the *modulation operator* M_ω on $L^2(G)$ as

$$(M_\omega f)(y) = f(y)\; \omega(y),$$

where $f \in L^2(G)$ and $y \in G$. In the next lemma, we list some properties of Gabor transform which can be verified easily, so we omit the proofs.

Lemma 3.1. Let $f, \psi \in L^2(G)$. For $x, z \in G$ and $\gamma, \omega \in \widehat{G}$, we have

(i) $G_\psi f(x,\gamma) = \gamma(x^{-1})\; \overline{G_f \psi(x^{-1}, \gamma^{-1})}$.

(ii) $G_\psi(M_\omega T_z f)(x,\gamma) = (\omega^{-1}\gamma)(z^{-1})\; G_\psi f(z^{-1}x, \omega^{-1}\gamma)$.

(iii) $G_{(M_\omega T_z \psi)}(M_\omega T_z f)(x,\gamma) = \omega(x)\; \gamma(z^{-1})\; G_\psi f(x,\gamma)$.

Lemma 3.2. Let $f, \psi \in L^2(G)$ and $F : G \times \widehat{G} \to \mathbb{C}$ be defined as

$$F(x,\gamma) = \gamma(x)\ G_\psi f(x,\gamma)\ G_\psi f(x^{-1},\gamma^{-1}),$$

where $(x,\gamma) \in G \times \widehat{G}$. For $(\omega, z) \in \widehat{G} \times G$, we have $\widehat{F}(\omega,z) = F(z^{-1},\omega)$.

Proof. For $z \in G$ and $\omega \in \widehat{G}$, by Lemma 3.1, we have

$$\begin{aligned}
\widehat{F}(\omega,z) &= \int_G \int_{\widehat{G}} F(x,\gamma)\ \omega(x^{-1})\ \gamma(z^{-1})\ dx\ d\gamma \\
&= \int_G \int_{\widehat{G}} \gamma(x)\ G_\psi f(x,\gamma)\ \gamma^{-1}(x)\ \overline{G_f\psi(x,\gamma)}\ \omega(x^{-1})\ \gamma(z^{-1})\ dx\ d\gamma \\
&= \int_G \int_{\widehat{G}} (G_\psi f\ \overline{G_f\psi})(x,\gamma)\ \omega(x^{-1})\ \gamma(z^{-1})\ dx\ d\gamma \\
&= \int_G \int_{\widehat{G}} G_\psi f(x,\gamma)\ \overline{G_f\psi(x,\gamma)\ \omega(x)\ \gamma(z)}\ dx\ d\gamma \\
&= \int_G \int_{\widehat{G}} G_\psi f(x,\gamma)\ \overline{G_{(M_\omega T_{z^{-1}} f)}(M_\omega T_{z^{-1}}\psi)(x,\gamma)}\ dx\ d\gamma \\
&= \langle G_\psi f, G_{(M_\omega T_{z^{-1}} f)}(M_\omega T_{z^{-1}}\psi)\rangle \\
&= \langle f, M_\omega T_{z^{-1}}\psi\rangle\ \overline{\langle \psi, M_\omega T_{z^{-1}} f\rangle} \\
&= G_\psi f(z^{-1},\omega)\ \overline{G_f\psi(z^{-1},\omega)} \\
&= (G_\psi f\ \overline{G_f\psi})(z^{-1},\omega) = F(z^{-1},\omega).
\end{aligned}$$

□

The structure theory of locally compact abelian groups suggests that G decomposes into a direct product $G = \mathbb{R}^n \times H$, where $n \geq 0$ and H contains a compact open subgroup. So the connected component of identity of G in non-compact if and only if $n \geq 1$. Let G be a locally compact abelian group such that the connected component of identity is non-compact. We can write $G = \mathbb{R} \times K$, where $K = \mathbb{R}^{n-1} \times H$, and $\widehat{G} = \widehat{\mathbb{R}} \times \widehat{K}$. We now prove the following analogue of Hardy's theorem for Gabor transform:

Theorem 3.3. Let G be a locally compact abelian group such that the connected component of identity is non-compact. Let $f, \psi \in L^2(G)$ such that

$$|G_\psi f(x,k,\xi,\gamma)| \leq e^{-\pi(ax^2+b\xi^2)/2}\ \varphi(k)\ \eta(\gamma),$$

for all $(x,k) \in G = \mathbb{R} \times K$, $(\xi,\gamma) \in \widehat{G} = \widehat{\mathbb{R}} \times \widehat{K}$, where a, b are positive real numbers; φ and η are bounded functions in $L^2(K)$ and $L^2(\widehat{K})$ respectively. If $ab > 1$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. For $(x,k),(z,t) \in \mathbb{R} \times K$ and $(\xi,\gamma),(\zeta,\chi) \in \widehat{\mathbb{R}} \times \widehat{K}$, we define

$$\begin{aligned}
F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma) = e^{2\pi i\xi x}\ &\gamma(k)\ G_\psi(M_{\zeta,\chi}T_{z,t}f)(x,k,\xi,\gamma) \\
&\times G_\psi(M_{\zeta,\chi}T_{z,t}f)(-x,k^{-1},-\xi,\gamma^{-1}).
\end{aligned}$$

The function $F_{(z,t,\zeta,\chi)}$ is continuous and is in $L^1(\mathbb{R} \times K \times \widehat{\mathbb{R}} \times \widehat{K})$. By Lemma 3.2, we have

$$\widehat{F_{(z,t,\zeta,\chi)}}(\omega, \delta, y, v) = F_{(z,t,\zeta,\chi)}(-y, v^{-1}, \omega, \delta).$$

Using Lemma 3.1(ii), we have

$$\begin{aligned}
&F_{(z,t,\zeta,\chi)}(x, k, \xi, \gamma) \\
&= e^{2\pi i \xi x} \ \gamma(k) \ e^{-2\pi i(\xi-\zeta)z} \ (\chi^{-1}\gamma)(t^{-1}) \ G_\psi f(x-z, t^{-1}k, \xi-\zeta, \chi^{-1}\gamma) \\
&\qquad \times e^{-2\pi i(-\xi-\zeta)z} \ (\chi^{-1}\gamma^{-1})(t^{-1}) \ G_\psi f(-x-z, t^{-1}k^{-1}, -\xi-\zeta, \chi^{-1}\gamma^{-1}),
\end{aligned}$$

which implies

$$\begin{aligned}
&|F_{(z,t,\zeta,\chi)}(x, k, \xi, \gamma)| \\
&= |G_\psi f(x-z, t^{-1}k, \xi-\zeta, \chi^{-1}\gamma)| \ |G_\psi f(-x-z, t^{-1}k^{-1}, -\xi-\zeta, \chi^{-1}\gamma^{-1})| \\
&\le e^{-\pi[a(x-z)^2+b(\xi-\zeta)^2]/2} \ \varphi(t^{-1}k) \ \eta(\chi^{-1}\gamma) \ e^{-\pi[a(-x-z)^2+b(-\xi-\zeta)^2]/2} \ \varphi(t^{-1}k^{-1}) \ \eta(\chi^{-1}\gamma^{-1}) \\
&= e^{-\pi a(x^2+z^2)} \ e^{-\pi b(\xi^2+\zeta^2)} \ \varphi(t^{-1}k) \ \eta(\chi^{-1}\gamma) \ \varphi(t^{-1}k^{-1}) \ \eta(\chi^{-1}\gamma^{-1}) \\
&= e^{-\pi a x^2} \ \beta_{(z,t,\zeta,\chi)}(k, \xi, \gamma),
\end{aligned}$$

where $\beta_{(z,t,\zeta,\chi)}$ is a function on $K \times \widehat{\mathbb{R}} \times \widehat{K} = S$ (say) given by

$$\beta_{(z,t,\zeta,\chi)}(k, \xi, \gamma) = e^{-\pi a z^2} \ e^{-\pi b(\xi^2+\zeta^2)} \ \varphi(t^{-1}k) \ \eta(\chi^{-1}\gamma) \ \varphi(t^{-1}k^{-1}) \ \eta(\chi^{-1}\gamma^{-1}).$$

Using Cauchy-Schwarz inequality, it follows that $\beta_{(z,t,\zeta,\chi)} \in L^1(S) \cap L^2(S)$.

$$\begin{aligned}
\text{Also, } |\widehat{F_{(z,t,\zeta,\chi)}}(\omega, \delta, y, v)| &= |F_{(z,t,\zeta,\chi)}(-y, v^{-1}, \omega, \delta)| \\
&\le e^{-\pi a(y^2+z^2)} \ e^{-\pi b(\omega^2+\zeta^2)} \ \varphi(t^{-1}v^{-1}) \ \eta(\chi^{-1}\delta) \ \varphi(t^{-1}v) \ \eta(\chi^{-1}\delta^{-1}) \\
&= e^{-\pi b\omega^2} \ \rho_{(z,t,\zeta,\chi)}(\delta, y, v),
\end{aligned}$$

where $\rho_{(z,t,\zeta,\chi)}$ is a function on $\widehat{S} = \widehat{K} \times \mathbb{R} \times K$ given by

$$\rho_{(z,t,\zeta,\chi)}(\delta, y, v) = e^{-\pi a(y^2+z^2)} \ e^{-\pi b\zeta^2} \ \varphi(t^{-1}v^{-1}) \ \eta(\chi^{-1}\delta) \ \varphi(t^{-1}v) \ \eta(\chi^{-1}\delta^{-1}).$$

Again, using Cauchy-Schwarz inequality, we have $\rho_{(z,t,\zeta,\chi)} \in L^1(\widehat{S}) \cap L^2(\widehat{S})$ and is also bounded.
Thus, by [1, Theorem 1.1], we have $F_{(z,t,\zeta,\chi)} \equiv 0$ for all (z, t, ζ, χ), whenever $ab > 1$.
Since, $F_{(-z,t^{-1},-\zeta,\chi^{-1})}(0, e, 0, I) = e^{4\pi i\zeta z} \ \chi(t)^2 \ G_\psi f(z, t, \zeta, \chi)$.
So, $G_\psi f \equiv 0$, whenever $ab > 1$.
By (2.2), we have $\|\psi\|_2 \ \|f\|_2 = 0$ which implies either $f = 0$ a.e. or $\psi = 0$ a.e., whenever $ab > 1$. $\square$

4. Euclidean Motion Group $M(n)$

Consider the Euclidean motion group $M(n) = \mathbb{R}^n \rtimes SO(n)$ with the group law given by

$$(z, k)(w, k') = (z + k \cdot w, kk'),$$

for $z, w \in \mathbb{R}^n$ and $k, k' \in SO(n)$. Let $K = SO(n)$, then $M = SO(n-1)$ can be considered as a subgroup of K leaving the point $e_1 = (1, 0, 0, \dots, 0)$ fixed. We retain the notations of [16]. Let $\widehat{M}$ denote the unitary dual of M. All the irreducible unitary representations of $M(n)$ relevant for the Plancherel formula are parametrized (upto unitary equivalence) by pairs (λ, σ), where $\lambda > 0$ and $\sigma \in \widehat{M}$.

Given $\sigma \in \widehat{M}$ realized on a Hilbert space H_σ of dimension d_σ, consider the space

$$L^2(K, \sigma) = \Big\{\varphi \mid \varphi : K \to M_{d_\sigma \times d_\sigma}, \int \|\varphi(k)\|^2 \, dk < \infty,$$
$$\varphi(uk) = \sigma(u)\varphi(k), \text{ for } u \in M \text{ and } k \in K\Big\}.$$

Note that $L^2(K, \sigma)$ is a Hilbert space under the inner product

$$\langle \varphi, \psi \rangle = \int_K tr(\varphi(k)\psi(k)^*) \, dk.$$

For each $\lambda > 0$ and $\sigma \in \widehat{M}$, we can define a representation $\pi_{\lambda,\sigma}$ of $M(n)$ on $L^2(K, \sigma)$ as follows:

For $\varphi \in L^2(K, \sigma)$, $(z, k) \in M(n)$,

$$\pi_{\lambda,\sigma}(z, k)\varphi(u) = e^{i\lambda\langle u^{-1} \cdot e_1, z\rangle} \, \varphi(uk),$$

for $u \in K$.

If $\varphi_j(k)$ are the column vectors of $\varphi \in L^2(K, \sigma)$, then $\varphi_j(uk) = \sigma(u)\varphi_j(k)$ for all $u \in M$. Therefore, $L^2(K, \sigma)$ can be written as the direct sum of d_σ copies of $H(K, \sigma)$, where

$$H(K, \sigma) = \Big\{\varphi \mid \varphi : K \to \mathbb{C}^{d_\sigma}, \int \|\varphi(k)\|^2 \, dk < \infty,$$
$$\varphi(uk) = \sigma(u)\varphi(k), \text{ for } u \in M \text{ and } k \in K\Big\}.$$

It can be shown that $\pi_{\lambda,\sigma}$ restricted to $H(K, \sigma)$ is an irreducible unitary representation of $M(n)$. Moreover, any irreducible unitary representation of $M(n)$ which is infinite dimensional is unitarily equivalent to one and only one $\pi_{\lambda,\sigma}$.

The Fourier transform of $f \in L^2(M(n))$ is given by,

$$\widehat{f}(\lambda, \sigma) = \int_{M(n)} f(z, k) \, \pi_{\lambda,\sigma}(z, k)^* \, dz \, dk.$$

$\widehat{f}(\lambda, \sigma)$ is a Hilbert-Schmidt operator on $H(K, \sigma)$.

A solid harmonic of degree m is a polynomial which is homogeneous of degree m and whose Laplacian is zero. The set of all such polynomials will be denoted by $\mathbb{H}_m$ and the restrictions of elements of $\mathbb{H}_m$ to S^{n-1} is denoted by S_m. By choosing an orthonormal basis $\{g_{mj} : j = 1, 2, \ldots, d_m\}$ of S_m for each $m = 0, 1, 2, \ldots$, we get an orthonormal basis for $L^2(S^{n-1})$.

The Haar measure on $M(n)$ is $dg = dz\ dk$, where dz is Lebesgue measure on $\mathbb{R}^n$ and dk is the normalized Haar measure on $SO(n)$. We now prove the following analogue of Hardy's theorem for Gabor transform on Euclidean motion group:

THEOREM 4.1. Let $f, \psi \in L^2(M(n))$ be such that

$$\|G_\psi f(x, k, \lambda, \sigma)\|_{\mathrm{HS}} \leq C e^{-\pi(a\|x\|^2 + b\lambda^2)/2} \tag{4.1}$$

for all $(x, k) \in M(n)$ and $(\lambda, \sigma) \in \mathbb{R}_+ \times \widehat{M}$, where a, b and C are positive real numbers. If $ab > 1$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n$. For each $v \in \mathbb{R}^n$ and $\sigma \in \widehat{M}$, we define $F_{v,\sigma} : M(n) \to \mathbb{C}$ such that

$$F_{v,\sigma}(x, k) = \int_{\mathbb{R}^n} \|G_\psi f(x - v, k, \|\mu\|, \sigma)\|_{\mathrm{HS}}\ \|G_\psi f(-x - v, k, \|\mu\|, \sigma)\|_{\mathrm{HS}} \times \exp\,[-2\pi i \sum_{p=1}^{n} \mu_p x_p]\ d\mu.$$

We have

$$\begin{aligned}
|F_{v,\sigma}(x, k)| &\leq \int_{\mathbb{R}^n} \|G_\psi f(x - v, k, \|\mu\|, \sigma)\|_{\mathrm{HS}}\ \|G_\psi f(-x - v, k, \|\mu\|, \sigma)\|_{\mathrm{HS}}\ d\mu \\
&\leq C^2 \int_{\mathbb{R}^n} e^{-\pi(a\|x-v\|^2 + b\|\mu\|^2)/2}\ e^{-\pi(a\|-x-v\|^2 + b\|\mu\|^2)/2}\ d\mu \\
&= C^2 e^{-\pi a(\|x-v\|^2 + \|-x-v\|^2)/2} \int_{\mathbb{R}^n} e^{-\pi b\|\mu\|^2}\ d\mu \\
&= C^2 e^{-\pi a\|x\|^2}\ e^{-\pi a\|v\|^2} \left(\frac{1}{\sqrt{b}}\right)^n \\
&\leq C^2 e^{-\pi a\|x\|^2} \left(\frac{1}{\sqrt{b}}\right)^n = C_1 e^{-\pi a\|x\|^2},\ \text{where } C_1 = C^2 \left(\frac{1}{\sqrt{b}}\right)^n.
\end{aligned}$$

Also,

$$\begin{aligned}
&\|\widehat{F_{v,\sigma}}(\lambda, \sigma)\|_{\mathrm{HS}}^2 \\
&= \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K |\widehat{F_{v,\sigma}}(\lambda, \sigma) g_{mj}(u)|^2\ du \\
&= \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K |\int_{M(n)} F_{v,\sigma}(x, k)\ \pi_{\lambda,\sigma}(x, k)^* g_{mj}(u)\ dx\ dk|^2\ du
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K | \int_{M(n)} \int_{\mathbb{R}^n} \|G_\psi f(x-v,k,\|\mu\|,\sigma)\|_{\mathrm{HS}} \ \|G_\psi f(-x-v,k,\|\mu\|,\sigma)\|_{\mathrm{HS}}$$

$$\exp{[-2\pi i \sum_{p=1}^{n} \mu_p x_p]} \ \pi_{\lambda,\sigma}(x,k)^* g_{mj}(u) \ d\mu \ dx \ dk|^2 \ du$$

$$\leq C^4 \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K | \int_{M(n)} \int_{\mathbb{R}^n} e^{-\pi(a\|x-v\|^2+b\|\mu\|^2)/2} \ e^{-\pi(a\|-x-v\|^2+b\|\mu\|^2)/2}$$

$$\exp{[-2\pi i \sum_{p=1}^{n} \mu_p x_p]} \ \pi_{\lambda,\sigma}(x,k)^* g_{mj}(u) \ d\mu \ dx \ dk|^2 \ du$$

$$= C^4 \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K | \int_{M(n)} \int_{\mathbb{R}^n} e^{-\pi a(\|x\|^2+\|v\|^2)} \ e^{-\pi b\|\mu\|^2}$$

$$\exp{[-2\pi i \sum_{p=1}^{n} \mu_p x_p]} \ \exp{[-i\lambda\langle u^{-1}\cdot e_1, x\rangle]} \ \overline{g_{mj}(uk)} \ d\mu \ dx \ dk|^2 \ du$$

$$= C^4 \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K | \int_{M(n)} \int_{\mathbb{R}^n} e^{-\pi a(\|x\|^2+\|v\|^2)} \ e^{-\pi b\|\mu\|^2}$$

$$\exp{[-2\pi i \sum_{p=1}^{n} \mu_p x_p]} \ \exp{[-i\lambda\langle u^{-1}\cdot e_1, x\rangle]} \ \overline{g_{mj}(k)} \ d\mu \ dx \ dk|^2 \ du$$

$$\leq C^4 \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K | \int_{M(n)} \int_{\mathbb{R}^n} e^{-\pi b\|\mu\|^2}$$

$$\exp{[-2\pi i \sum_{p=1}^{n} \mu_p x_p]} \ \exp{[-2\pi i\lambda\langle u^{-1}\cdot e_1, x\rangle]} \ \overline{g_{mj}(k)} \ d\mu \ dx \ dk|^2 \ du.$$

Let $u = [u_{ij}]_{n\times n} \in K$. Then

$$u^{-1}\cdot e_1 = u^T \cdot e_1 = [u_{11}, u_{12}, \ldots, u_{1n}]^T.$$

So, $\langle u^{-1}\cdot e_1, x\rangle = \sum\limits_{p=1}^{n} u_{1p} \ x_p$, thus

$$\|\widehat{F_{v,\sigma}}(\lambda,\sigma)\|_{\mathrm{HS}}^2$$

$$\leq C^4 \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K | \int_{M(n)} \int_{\mathbb{R}^n} \exp{[-\pi b \sum_{p=1}^{n} \mu_p^2]} \ \exp{[-2\pi i \sum_{p=1}^{n} \mu_p x_p]}$$

$$\exp{[-2\pi i\lambda \sum_{p=1}^{n} u_{1p} \ x_p]} \ \overline{g_{mj}(k)} \ d\mu \ dx \ dk|^2 \ du$$

$$= C^4 \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K |\int_K \int_{\mathbb{R}^n} \exp[-\frac{\pi}{b} \sum_{p=1}^{n} x_p^2] \ \exp[-2\pi i \lambda \sum_{p=1}^{n} u_{1p} \ x_p] \ \overline{g_{mj}(k)} \ dx \ dk|^2 \ du$$

$$= C^4 \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K |\int_K \exp[-\pi b \lambda^2 \sum_{p=1}^{n} u_{1p}^2] \ \overline{g_{mj}(k)} \ dk|^2 \ du$$

$$= C^4 \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K |\int_K e^{-\pi b \lambda^2} \ \overline{g_{mj}(k)} \ dk|^2 \ du$$

$$= C^4 e^{-2\pi b \lambda^2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \int_K |\int_K \overline{g_{mj}(k)} \ dk|^2 \ du$$

$$= C^4 e^{-2\pi b \lambda^2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} |\int_K \overline{g_{mj}(k)} \ dk|^2,$$

which implies

$$\|\widehat{F_{v,\sigma}}(\lambda, \sigma)\|_{\mathrm{HS}}^2 \leq C_2 e^{-\pi b \lambda^2},$$

where $C_2 = C^2$. Let $C = \max\{C_1, C_2\}$, then the function $F_{v,\sigma}$ satisfies the conditions of Hardy's Theorem on $M(n)$ (see [18]), if $ab > 1$, then $F_{v,\sigma}(x,k) = 0$ for all $(x,k) \in M(n)$, $v \in \mathbb{R}^n$ and $\sigma \in \widehat{M}$.
In particular, we have for all $v \in \mathbb{R}^n$, $k \in K$, $\sigma \in \widehat{M}$

$$F_{-v,\sigma}(0,k) = 0$$

$$\Rightarrow \quad \int_{\mathbb{R}^n} \|G_\psi f(v, k, \|\mu\|, \sigma)\|_{\mathrm{HS}}^2 \ d\mu = 0$$

$$\Rightarrow \quad \|G_\psi f(v, k, \|\mu\|, \sigma)\|_{\mathrm{HS}}^2 = 0$$

for all $\mu \in \mathbb{R}^n$. Using spherical polar coordinates, we can conclude that $\|G_\psi f(v,k,\lambda,\sigma)\|_{\mathrm{HS}}^2 = 0$ for almost all $\lambda > 0$. By Heisenberg uncertainty inequality for Gabor transform on $M(n)$ (see [4]) or (2.2) above, we have $\|\psi\|_2 \ \|f\|_2^2 = 0$ which implies either $f = 0$ a.e. or $\psi = 0$ a.e., whenever $ab > 1$. $\square$

5. A Class of Nilpotent Lie Groups

In this section, we shall prove Hardy's theorem for a class of connected, simply connected nilpotent Lie groups or connected nilpotent Lie groups with non-compact centre for which the Hilbert-Schmidt norm of the group Fourier transform $\pi_\xi(f)$ of f attains a particular form.
Let $G = \exp \mathfrak{g}$ be the associated connected and simply connected nilpotent Lie group of an n-dimensional real nilpotent Lie algebra $\mathfrak{g}$ (see [5]).
Let $\mathcal{B} = \{X_1, X_2, \ldots, X_n\}$ denote a strong Malcev basis of $\mathfrak{g}$ through the ascending central series of $\mathfrak{g}$. For $x = \exp(x_1X_1 + x_2X_2 + \ldots + x_nX_n) \in G$,

$x_j \in \mathbb{R}$, we define a *norm function* on G as

$$\|x\| = (x_1^2 + \ldots + x_n^2)^{1/2}.$$

The composed map

$$\mathbb{R}^n \to \mathfrak{g} \to G,$$

given as

$$(x_1, \ldots, x_n) \to \sum_{j=1}^{n} x_j X_j \to \exp\left(\sum_{j=1}^{n} x_j X_j\right),$$

is a diffeomorphism and maps Lebesgue measure on $\mathbb{R}^n$ to Haar measure on G. In this way, we shall always identify $\mathfrak{g}$, and sometimes G, as sets with $\mathbb{R}^n$. Thus, measurable (integrable) functions on G can be viewed as measurable (integrable) functions on $\mathbb{R}^n$.

Let $\mathfrak{g}^*$ denote the vector space dual of $\mathfrak{g}$ and $\{X_1^*, \ldots, X_n^*\}$ the basis of $\mathfrak{g}^*$ which is dual to $\{X_1, \ldots, X_n\}$. Then, $\{X_1^*, \ldots, X_n^*\}$ is a Jordan-Hölder basis for the coadjoint action of G on $\mathfrak{g}^*$. We shall identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ via the map

$$\xi = (\xi_1, \ldots, \xi_n) \to \sum_{j=1}^{n} \xi_j X_j^*.$$

We introduce the Euclidean norm on $\mathfrak{g}^*$ relative to the basis $\{X_1^*, \ldots, X_n^*\}$ as

$$\left\| \sum_{j=1}^{n} \xi_j X_j^* \right\| = (\xi_1^2 + \ldots + \xi_n^2) = \|\xi\|.$$

Let $\mathfrak{g}_j = \mathbb{R}\text{-span}\{X_1, \ldots, X_n\}$. For $\xi \in \mathfrak{g}^*$, $\mathcal{O}_\xi$ denotes the coadjoint orbit of ξ. An index $j \in \{1, 2, \ldots, n\}$ is a jump index for ξ if

$$\mathfrak{g}(\xi) + \mathfrak{g}_j \neq \mathfrak{g}(\xi) + \mathfrak{g}_{j-1}.$$

We consider the set

$$e(\xi) = \{j : j \text{ is a jump index for } \xi\},$$

which contains exactly $\dim(\mathcal{O}_l)$ indices. Also, there are two disjoint sets S and T of indices with $S \cup T = \{1, \ldots, n\}$ and a G-invariant Zariski open set $\mathscr{U}$ of $\mathfrak{g}^*$ such that $e(\xi) = S$ for all $\xi \in \mathscr{U}$. The Pfaffian $\mathrm{Pf}(\xi)$ of the skew-symmetric matrix $M_S(\xi) = (\xi([X_i, X_j]))_{i,j \in S}$ is defined as

$$|\operatorname{Pf}(\xi)|^2 = \det M_S(\xi).$$

Let $V_S = \mathbb{R}\text{-span}\{X_i^* : i \in S\}$, $V_T = \mathbb{R}\text{-span}\{X_i^* : i \in T\}$ and $d\xi$ be the Lebesgue measure on V_T such that the unit cube spanned by $\{X_i^* : i \in T\}$ has volume 1. Then, $\mathfrak{g}^* = V_T \oplus V_S$ and V_T meets $\mathscr{U}$. Let $\mathcal{W} = \mathscr{U} \cap V_T$ be the cross-section for the coadjoint orbits through the points in $\mathscr{U}$. If

$d\xi$ is the Lebesgue measure on $\mathcal{W}$, then $d\mu(\xi) = |\operatorname{Pf}(\xi)|\; d\xi$ is a Plancherel measure for $\hat{G}$. The Plancherel formula is given by

$$\|f\|_2^2 = \int_{\mathcal{W}} \|\pi_\xi(f)\|_{\mathrm{HS}}^2 \; d\mu(\xi), \quad f \in L^1 \cap L^2(G),$$

where $\|\pi_\xi(f)\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of $\pi_\xi(f)$ and dg is the Haar measure on G.
A careful reading of the proof of Hardy's theorem for connected, simply connected nilpotent Lie groups (see [10]) and for connected nilpotent Lie groups with non-compact centre (see [2]) yield the following:

Theorem 5.1. Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group or a connected nilpotent Lie group with non-compact centre. Let a, b and C be positive real numbers, and suppose that $f : G \to \mathbb{C}$ is a measurable function such that

(i) $|f(x)| \leq Ce^{-a\pi\|x\|^2}$, for all $x \in G$,
(ii) $|\operatorname{Pf}(\xi)|^{1/2}\; \|\pi_\xi(f)\|_{\mathrm{HS}} \leq Ce^{-b\pi\|\xi\|^2}$, for all $\xi \in \mathcal{W}$.

If $ab > 1$, then $f = 0$ a.e.

We shall consider the case in which $\mathcal{W}$ takes the following form:

$\mathcal{W} = \{\xi = (\xi_1, \ldots, \xi_n) \in \mathfrak{g}^* : \ \xi_j = 0 \text{ for } (n-k) \text{ values of } j \text{ with } |\operatorname{Pf}(\xi)| \neq 0\}.$

We denote the non-vanishing variables by $\xi_{p_1}, \xi_{p_2}, \ldots, \xi_{p_k}$, where $p_1 < p_2 < \ldots < p_k$. Without loss of generality, we may assume that $p_1 = 1$.
We denote the vanishing variables by $\xi_{m_1}, \xi_{m_2}, \ldots, \xi_{m_{n-k}}$, where $k > 0$ and $m_1 < m_2 < \ldots < m_{n-k}$.
Let $A = \{p_1, p_2, \ldots, p_k\}$ and $B = \{m_1, m_2, \ldots, m_{n-k}\}$ be the set of indices corresponding to the non-vanishing and vanishing variables respectively.
We consider the class of groups for which the Hilbert-Schmidt norm $\|\pi_\xi(f)\|_{\mathrm{HS}}$ for all $\xi \in \mathcal{W}$ and $f \in L^2(G)$ has the following form:

$$\|\pi_\xi(f)\|_{\mathrm{HS}}^2 = \frac{|h(\xi)|}{|\operatorname{Pf}(\xi)|} \int_{\mathbb{R}^{n-k}} |\mathscr{F}(f \circ \exp)\,(\xi_1 + Q_1, \ldots, \xi_n + Q_n)|^2 \; d\xi_{m_1} \; \ldots \; d\xi_{m_{n-k}}, \quad (*)$$

where $\mathscr{F}$ denotes the Fourier transform on $\mathbb{R}^k$, $h(\xi)$ is a polynomial in ξ and the functions $Q_1 = 0$, $Q_k = Q_k(\xi_1, \xi_2, \ldots, \xi_{k-1})$ with $2 \leq k \leq n$. Without loss of generality, we can assume that $Q_j = 0$ for all $j \in B$.
In this case, the continuous *Gabor Transform* of f with respect to the window function ψ can be defined as follows:

$$G_\psi f(x, \xi) := \int_G f(y)\; \overline{\psi(x^{-1}y)}\; \pi_\xi(y)^* \; dy,$$

where $(x, \xi) \in G \times \mathcal{W}$.
We now prove the following analogue of Hardy's theorem for Gabor transform on G:

THEOREM 5.2. Let G be a connected, simply connected nilpotent Lie group or a connected nilpotent Lie group with non-compact centre for which $\|\pi_\xi(f)\|_{\text{HS}}$ attains the form $(*)$ and let $f, \psi \in L^2(G)$ be such that

$$\|G_\psi f(x,\xi)\|_{\text{HS}} \leq C e^{-\pi(a\|x\|^2 + b\|\xi\|^2)/2} \tag{5.1}$$

for all $(x,\xi) \in G \times \mathcal{W}$, where a, b and C are positive real numbers. If $ab > 1$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. Let $z = \exp(\sum_{j=1}^{n} z_j X_j), v = \exp(\sum_{j=1}^{n} v_j X_j) \in G$, where $z_j, v_j \in \mathbb{R}$.
Let $Z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ and $V = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$.
Denote $\mu = \sum_{j \in A} \mu_j X_j^* \in \mathcal{W}$. Assuming $U = (\mu_1, \mu_2, \ldots, \mu_n)$ with $\mu_{m_i} = 0$ for $1 \leq i \leq n-k$, $Y = (y_1, y_2, \ldots, y_n)$ with $y_{m_i} = 0$ for $1 \leq i \leq n-k$, we can consider U and Y as elements of $\mathbb{R}^k$.
We define $F_v : G \to \mathbb{C}$ such that

$$\begin{aligned}
F_v(z) &= \int_{\mathcal{W}} \|G_\psi f(zv^{-1}, \mu)\|_{\text{HS}} \ \|G_\psi f(z^{-1}v^{-1}, \mu)\|_{\text{HS}} \\
&\qquad \times \exp\,[-2\pi i \sum_{j\in A} \mu_j z_j] \ \exp\,[2\pi i \sum_{j \in A} Q_j z_j] \ d\mu \\
&= \int_{\mathcal{W}} \|G_\psi f(\exp\,(\sum_{j=1}^{n} (z_j - v_j) X_j), \mu)\|_{\text{HS}} \ \|G_\psi f(\exp\,(\sum_{j=1}^{n} (-z_j - v_j) X_j), \mu)\|_{\text{HS}} \\
&\qquad \times \exp\,[-2\pi i \sum_{j\in A} \mu_j z_j] \ \exp\,[2\pi i \sum_{j \in A} Q_j z_j] \ d\mu.
\end{aligned}$$

We have

$$\begin{aligned}
|F_v(z)| &\leq \int_{\mathcal{W}} \|G_\psi f(zv^{-1}, \mu)\|_{\text{HS}} \ \|G_\psi f(z^{-1}v^{-1}, \mu)\|_{\text{HS}} \ d\mu \\
&\leq C^2 \int_{\mathcal{W}} e^{-\pi(a\|zv^{-1}\|^2 + b\|\mu\|^2)/2} \ e^{-\pi(a\|z^{-1}v^{-1}\|^2 + b\|\mu\|^2)/2} \ d\mu \\
&= C^2 \int_{\mathbb{R}^k} e^{-\pi(a\|Z-V\|^2 + b\|U\|^2)/2} \ e^{-\pi(a\|-Z-V\|^2 + b\|U\|^2)/2} \ dU \\
&= C^2 e^{-\pi a(\|Z-V\|^2 + \|-Z-V\|^2)/2} \int_{\mathbb{R}^k} e^{-\pi b\|U\|^2} \ dU \\
&= C^2 e^{-\pi a(\|Z\|^2 + \|V\|^2)} \int_{\mathbb{R}^k} e^{-\pi b\|U\|^2} \ dU \\
&= C^2 e^{-\pi a\|z\|^2} \ e^{-\pi a\|v\|^2} \ \left(\frac{1}{b}\right)^{k/2} \\
&\leq C^2 e^{-\pi a\|z\|^2} \ \left(\frac{1}{b}\right)^{k/2} = C_1 e^{-\pi a\|z\|^2}, \text{ where } C_1 = C^2 \left(\frac{1}{b}\right)^{k/2}.
\end{aligned}$$

$$
\begin{aligned}
&|\operatorname{Pf}(\xi)|\ \|\pi_\xi(F_v)\|_{\mathrm{HS}}^2\\
&= |h(\xi)| \int_{\mathbb{R}^{n-k}} |\mathscr{F}(F_v \circ \exp)(\xi_1+Q_1,\dots,\xi_n+Q_n)|^2\ d\xi_{m_1}\ \dots\ d\xi_{m_{n-k}}\\
&= |h(\xi)| \int_{\mathbb{R}^{n-k}} \Big| \int_{\mathbb{R}^k} F_v(\exp(\sum_{j\in A} y_j X_j + \sum_{j\in B} \xi_j X_j))\\
&\qquad \times \exp[-2\pi i \sum_{j\in A} y_j(\xi_j+Q_j)]\ dy_{p_1}\ dy_{p_2}\ \dots\ dy_{p_k}\Big|^2\ d\xi_{m_1}\ \dots\ d\xi_{m_{n-k}}\\
&= |h(\xi)| \int_{\mathbb{R}^{n-k}} \Big| \int_{\mathbb{R}^k}\int_{\mathcal{W}} \|G_\psi f(\exp(\sum_{j\in A}(y_j-v_j)X_j + \sum_{j\in B}(\xi_j-v_j)X_j),\mu)\|_{\mathrm{HS}}\\
&\qquad \times \|G_\psi f(\exp(\sum_{j\in A}(-y_j-v_j)X_j + \sum_{j\in B}(-\xi_j-v_j)X_j),\mu)\|_{\mathrm{HS}}\\
&\qquad \times \exp[-2\pi i\sum_{j\in A}\mu_j y_j]\ \exp[-2\pi i \sum_{j\in A} y_j\xi_j]\ d\mu\ dy_{p_1}\ dy_{p_2}\ \dots\ dy_{p_k}\Big|^2\ d\xi_{m_1}\ \dots\ d\xi_{m_{n-k}}\\
&\le C^2|h(\xi)| \int_{\mathbb{R}^{n-k}} \Big| \int_{\mathbb{R}^k}\int_{\mathcal{W}} \exp[-\frac{\pi a}{2}(\sum_{j\in A}(y_j-v_j)^2 + \sum_{j\in B}(\xi_j-v_j)^2)]\ e^{-\pi b\|\mu\|^2/2}\\
&\qquad \times \exp[-\frac{\pi a}{2}(\sum_{j\in A}(-y_j-v_j)^2 + \sum_{j\in B}(-\xi_j-v_j)^2)]\ e^{-\pi b\|\mu\|^2/2}\\
&\qquad \times \exp[-2\pi i\sum_{j\in A}\mu_j y_j]\ \exp[-2\pi i \sum_{j\in A} y_j\xi_j]\ d\mu\ dy_{p_1}\ dy_{p_2}\ \dots\ dy_{p_k}\Big|^2\ d\xi_{m_1}\ \dots\ d\xi_{m_{n-k}}\\
&= C^2|h(\xi)| \int_{\mathbb{R}^{n-k}} \Big| \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \exp[-\pi a(\sum_{j\in A}(y_j^2+v_j^2) + \sum_{j\in B}(\xi_j^2+v_j^2))]\ e^{-\pi b\|U\|_2^2}\\
&\qquad \times \exp[-2\pi i\sum_{j\in A}\mu_j y_j]\ \exp[-2\pi i \sum_{j\in A} y_j\xi_j]\ dU\ dy_{p_1}\ dy_{p_2}\ \dots\ dy_{p_k}\Big|^2\ d\xi_{m_1}\ \dots\ d\xi_{m_{n-k}}\\
&\le |h(\xi)| \int_{\mathbb{R}^{n-k}} \Big| \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \exp[-\pi a\sum_{j\in B}\xi_j^2]\ e^{-\pi b\|U\|_2^2}\ \exp[-2\pi i\sum_{j\in A}\mu_j y_j]\\
&\qquad \times \exp[-2\pi i \sum_{j\in A} y_j\xi_j]\ dU\ dy_{p_1}\ dy_{p_2}\ \dots\ dy_{p_k}\Big|^2\ d\xi_{m_1}\ \dots\ d\xi_{m_{n-k}}\\
&= C^2|h(\xi)| \int_{\mathbb{R}^{n-k}} \Big| \int_{\mathbb{R}^k} \exp[-\pi a\sum_{j\in B}\xi_j^2]\ \left(\frac{1}{b}\right)^{k/2}\ e^{-\pi\|Y\|_2^2/b}\\
&\qquad \times \exp[-2\pi i \sum_{j\in A} y_j\xi_j]\ dy_{p_1}\ dy_{p_2}\ \dots\ dy_{p_k}\Big|^2\ d\xi_{m_1}\ \dots\ d\xi_{m_{n-k}}\\
&= C^2|h(\xi)| \left(\frac{1}{b}\right)^k \int_{\mathbb{R}^{n-k}} \exp[-2\pi a\sum_{j\in B}\xi_j^2]\ d\xi_{m_1}\ \dots\ d\xi_{m_{n-k}}\\
&\qquad \times \Big| \int_{\mathbb{R}^k} e^{-\pi\|Y\|_2^2/b}\ \exp[-2\pi i\sum_{j\in A} y_j\xi_j]\ dy_{p_1}\ dy_{p_2}\ \dots\ dy_{p_k}\Big|^2
\end{aligned}
$$

$$= C^2|h(\xi)| \left(\frac{1}{b}\right)^k \left(\frac{1}{2a}\right)^{(n-k)/2} b^k\ e^{-2b\pi\|\xi\|^2}$$
$$= C^2|h(\xi)| \left(\frac{1}{2a}\right)^{(n-k)/2} e^{-2b\pi\|\xi\|^2}.$$

Now assuming that $ab > 1$, choose $0 < \gamma < b$ such that $a\gamma > 1$. Then, since $h(\xi)$ is a polynomial function, there exists a constant $K > 0$ such that

$$|h(\xi)|\ e^{-2(b-\gamma)\pi\|\xi\|^2} \leq K$$

for all $\xi \in \mathcal{W}$. So

$$\begin{aligned}|\operatorname{Pf}(\xi)|\ \|\pi_\xi(F_v)\|_{\mathrm{HS}}^2 &\leq KC^2 \left(\frac{1}{2a}\right)^{(n-k)/2} e^{2(b-\gamma)\pi\|\xi\|^2}\ e^{-2b\pi\|\xi\|^2} \\ &= C_2^2\ e^{-2\gamma\pi\|\xi\|^2},\ \text{where } C_2^2 = KC^2\left(\frac{1}{2a}\right)^{(n-k)/2}.\end{aligned}$$

Let $C = \max\{C_1, C_2\}$, then the function F_v satisfies the conditions of Theorem 5.1, if $a\gamma > 1$, then $F_v(z) = 0$ for all $z, v \in G$.
In particular, we have for all $v \in G$

$$\begin{aligned} & F_{v^{-1}}(1) = 0 \\ \Rightarrow \quad & \int_{\mathcal{W}} \|G_\psi f(v,\mu)\|_{\mathrm{HS}}^2\ d\mu = 0 \\ \Rightarrow \quad & \|G_\psi f(v,\mu)\|_{\mathrm{HS}}^2 = 0 \end{aligned}$$

for all $\mu \in \mathcal{W}$. By Heisenberg uncertainty inequality for Gabor transform on G (see [4]) or (2.2) above, we have $\|\psi\|_2\ \|f\|_2^2 = 0$ which implies either $f = 0$ a.e. or $\psi = 0$ a.e., whenever $ab > 1$. □

Example 5.3. We now list several classes that are included in the above general class.

1. For thread-like nilpotent Lie groups (see [10]), we have $\operatorname{Pf}(\xi) = \xi_1$ and $\mathcal{W} = \{\xi = (\xi_1, 0, \xi_3, \ldots, \xi_{n-1}, 0) : \xi_j \in \mathbb{R}, \xi_1 \neq 0\}$. Also, $\|\pi_\xi(f)\|_{\mathrm{HS}}$ is given by

$$\|\pi_\xi(f)\|_{\mathrm{HS}}^2 = \frac{1}{|\operatorname{Pf}(\xi)|}\int_{\mathbb{R}^2} |\mathscr{F}(f\circ\exp)\,(\xi_1, t, \xi_3 + Q_3, \ldots, \xi_{n-1} + Q_{n-1}, s)|^2\ ds\ dt,$$

   where $Q_j(\xi_1, 0, \xi_3, \ldots, \xi_{j-1}, t) = \sum_{k=1}^{j-1} \frac{1}{k!}\ \frac{t^k}{\xi_1^k}\ \xi_{j-k}$, for $3 \leq j \leq n-1$. In particular, for Heisenberg groups.
2. For 2-NPC nilpotent Lie groups (see [3]), let $\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ be a Jordan-Hölder sequence in $\mathfrak{g}$ such that $\mathfrak{g}_m = \mathfrak{z}(g)$ and $\mathfrak{h} = \mathfrak{g}_{n-2}$. Let us consider the ideal $[\mathfrak{g}, \mathfrak{g}_{m+1}]$ of $\mathfrak{g}$ which is one or two dimensional in $\mathfrak{g}$. We discuss the two cases separately:
   (a) $\dim[\mathfrak{g}, \mathfrak{g}_{m+1}] = 2$.
   In this case, for every basis $\{X_1, X_2\}$ of $\mathfrak{h}$ in $\mathfrak{g}$ and $Y_1 \in \mathfrak{g}_{m+1} \setminus \mathfrak{z}(\mathfrak{g})$, the vectors $Z_1 = [X_1, Y_1]$ and $Z_2 = [X_2, Y_1]$ are linearly independent

and lie in the center of $\mathfrak{g}$.

Assume that $\mathfrak{g}_1 = \mathbb{R}\text{-span}\{Z_1\}$, $\mathfrak{g}_2 = \mathbb{R}\text{-span}\{Z_1, Z_2\}$. Let $Z_3, \ldots, Z_m$ be some vectors such that $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\text{-span}\{Z_1, \ldots, Z_m\}$ and $\mathcal{B} = \{Z_1, \ldots, Z_n\}$ a Jordan-Hölder basis of $\mathfrak{g}$ chosen as follows:

(i) $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\text{-span}\{Z_1, \ldots, Z_m\}$

(ii) $\mathfrak{h} = \mathbb{R}\text{-span}\{Z_1, \ldots, Z_{n-2}\}$

(iii) $\mathfrak{g} = \mathbb{R}\text{-span}\{Z_1, \ldots, Z_{n-2}, X_1 = Z_{n-1}, X_2 = Z_n\}$.

For $m_1 = m+1$ and $m+2 \le m_2 \le n-2$, we denote $Z_{m_1} = Z_{m+1} = Y_1$, $Z_{m_2} = Y_2$. These vectors can be chosen such that $\xi_1 = \xi([X_1, Y_1]) \neq 0$, $\xi_{2,2} = \xi([X_2, Y_2]) \neq 0$, for all $\xi \in \mathcal{W}$, where

$$\mathcal{W} = \{\xi = (\xi_1, \xi_2, \ldots, \xi_m, 0, 0, \xi_{m+3}, \xi_{m+4}, \ldots, \xi_{n-2}, 0, 0) : \xi_j \in \mathbb{R}, |\operatorname{Pf}(\xi)| \neq 0\}.$$

Also, we have $\operatorname{Pf}(\xi) = \xi(Z_1)\ \xi([X_2, Y_2]) - \xi([X_1, Y_2])\ \xi(Z_2)$ and $\|\pi_\xi(f)\|_{\mathrm{HS}}$ is given by

$$\|\pi_\xi(f)\|_{\mathrm{HS}}^2 = \frac{|h(\xi)|}{|\operatorname{Pf}(\xi)|} \times \int_{\mathbb{R}^4} |\mathscr{F}(f \circ \exp)\,(s_2, s_1, \xi_{n-2} + Q_{n-2}(\xi, t), \ldots, \xi_1 + Q_1(\xi, t))|^2\ ds_1\ ds_2\ dt,$$

where h is the polynomial defined by $h(\xi) = |\xi_1 \xi_{2,2}|$ and $Q_j(\xi, t)$ is a polynomial function with respect to the variables $t = (t_1, t_2)$ and $\xi_{m+1}, \ldots, \xi_{j-1}$ and rational in the variables $\xi_1, \ldots, \xi_m$.

(b) $\dim[\mathfrak{g}, \mathfrak{g}_{m+1}] = 1$.

In this case, we have $\operatorname{Pf}(\xi) = \xi([X_1, Y_1]) \cdot \xi([X_2, Y_2])$ and

$$\mathcal{W} = \{\xi = (\xi_1, \xi_2, \ldots, \xi_m, 0, \xi_{m+2}, \ldots, \xi_{m+d+1}, 0, \xi_{m+d+3}, \ldots, \xi_{n-2}, 0, 0) : \xi_j \in \mathbb{R}, |\operatorname{Pf}(\xi)| \neq 0\}.$$

Also, $\|\pi_\xi(f)\|_{\mathrm{HS}}$ is given by

$$\|\pi_\xi(f)\|_{\mathrm{HS}}^2 = \frac{1}{|\operatorname{Pf}(\xi)|} \int_{\mathbb{R}^4} \left| \mathscr{F}(f \circ \exp) \left(s_2, s_1, P_{n-2}\left(\xi, -\frac{t_1}{\xi_1}, -\frac{t_2 + R(-\frac{t_1}{\xi_1}, \xi_1, \ldots, \xi_{m+d})}{\xi_{2,2}}\right), \ldots, t_2, \ldots, P_{m+2}\left(\xi, -\frac{t_1}{\xi_1}\right), t_1, \xi_m, \ldots, \xi_1 \right) \right|^2 ds_1\ ds_2\ dt_1\ dt_2.$$

3. For low-dimensional nilpotent Lie groups of dimension at most 6 (see [12]) except for $G_{5,4}$, $G_{6,8}$, $G_{6,12}$, $G_{6,14}$, $G_{6,15}$, $G_{6,17}$, an explicit form of $\|\pi_\xi(f)\|_{\mathrm{HS}}$ can be obtained which is of the form $(*)$.

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